Variance Estimation for Calibration to Estimated Control Totals

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Tuesday, 11/05/2013
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What is calibration weighting?

Let \( \{ d_k \} \) be original survey (design) weights.

\( t_x = \sum_U x_k \) is a known total in the population with indices \( U \); \( x_k \) can be a vector.

The calibrated weights \( \{ w_k \} \) are “close” to \( \{ d_k \} \) but satisfy a set of calibration equations:

\[
\sum_s w_k x_k = \sum_U x_k.
\]

The “closeness” is measured by a distance function:

1. Linear calibration: \( E_p \{ \sum_s (w_k - d_k)^2 / d_k q_k \} \)
2. Raking: \( E_p \{ \sum_s [w_k \log(w_k/d_k) - w_k + d_k] \} \)
3. Logit method and others.

The calibration estimator \( \hat{t}_{yw} = \sum_s w_k y_k \).
Calibration Estimator Properties

- The calibration weights can be written as \( w_k = d_k F_k(x'_k \hat{\lambda}_s) \), where \( F_k(\cdot) \) comes from the inverse of distance function.
- The linear calibration estimator

\[
\hat{t}_{yl} = \sum_s w_k y_k = \hat{t}_A y + (t_x - \hat{t}_A x)' \hat{B}_s
\]

- \( AV(\hat{t}_{yw}) = AV(\hat{t}_{yl}) \).
- Calibration weighting can reduce mean squared error.
- There are various ways to compute the weights, including in the survey and TeachingSampling packages in R.
Calibration on Estimated Control Totals

- In calibration only $t_x$ are needed from outside source.
- Sometimes $t_x$ are not available and one may seek estimated totals $\hat{t}_{Cx}$ instead.
- The calibration constraint equation becomes:
  \[ \sum_s w_k x_k = \hat{t}_{Cx}. \]
- Calibration estimator with estimated control totals $\hat{t}_{yW}$ (CEEC).
- Assuming Analytic Survey and Control Total Survey are independent.
The linear CEEC

\[ \hat{t}_yL = \sum_s w_k y_k = \hat{t}_{Ay} + (\hat{t}_{Cx} - \hat{t}_{Ax})' \hat{B}_s \]

, where \( \hat{B}_s = T_s^{-1} \sum_s d_k q_k x_k y_k \), and \( T_s = \sum_s d_k q_k x_k x_k' \).

The general CEEC

\[ \hat{t}_yW = \sum_s w_k y_k = \sum_s d_k F_k (x_k' \hat{\gamma}_s) y_k \]

where \( \hat{\gamma}_s \) can be computationally solved from calibration constraints.
Assumptions for the CEEC

1. $\max ||x_k||$ and $F''_k(0)$ are both uniformly bounded by $K_1 < \infty$ and $K_2 < \infty$, respectively.

2. $\lim N^{-1} t_x$ exists.

3. $N^{-1}(\hat{t}_{Ax} - t_x) \rightarrow 0$ in design probability.

4. $n^{1/2} N^{-1}(\hat{t}_{Ax} - t_x) \xrightarrow{d} \text{MVN}(0, \Sigma_A)$.

5. $C_\lambda = \bigcap_{k \in U} \{ \lambda : x'_k \lambda \in \text{Im}_k(d_k) \}$ is a convex domain as well as an open neighborhood of 0.

6. $N^{-1}(\hat{t}_{Cx} - t_x) \rightarrow 0$ in design probability.

7. $n^\alpha N^{-1}(\hat{t}_{Cx} - t_x) \xrightarrow{d} \text{MVN}(0, \Sigma_C)$ with $\alpha \geq 1/2$, so the control total survey is at least as accurate as the analytic survey we conduct.
Properties of the CEEC

- $\hat{t}_{yL}$ and $\hat{t}_{yI}$ has exactly the same expectation.

- $N^{-1}(\hat{t}_{yL} - \hat{t}_{yI}) = O_p(n^{-\alpha})$

- $\hat{t}_{yW}$ is design consistent as well as asymptotically design unbiased.

- $N^{-1}(\hat{t}_{yW} - \hat{t}_{Ay}) = O_p(n^{-1/2})$ and $N^{-1}(\hat{t}_{yW} - \hat{t}_{yL}) = O_p(n^{-1})$.

- $n^{1/2}N^{-1}(\hat{t}_{yW} - \hat{t}_{yI}) = O_p(n^{1/2-\alpha})$. 
Variance Estimation for the CEEC

- Extra variation brought by the estimated control totals.
- Variance might be underestimated by traditional methods.
- The contribution of the bias to the MSE is likely to be small.
- The key is to precisely estimate the inflation part of the variance.

The asymptotic variance of $\hat{t}_{yw}$ is:

$$AV(\hat{t}_{yw}) = AV(\hat{t}_{yw}) + B' V(\hat{t}_{Cx}) B$$

$$= \sum \sum U \Delta_{kl} d_k E_k d_l E_l + B' V(\hat{t}_{Cx}) B,$$

where $B = (\sum_U x_k x'_k)^{-1}(\sum_U x_k y_k)$. 
A traditional variance estimator (Naive)

- Traditional variance estimator considers the estimated control totals as they were true population totals.
- Deville and Särndal, 1992, *JASA* first defined the calibration estimator as well as gave a variance estimator.
- The formula is:

\[ \hat{V}_{\text{naive}}(\hat{t}_yW) = \sum \sum_s (\Delta_{kl}/\pi_{kl})(w_k e_k)(w_l e_l) \]

where \( w_k = d_k F_k(x'_k \hat{\gamma}_s) \), and \( e_k = y_k - x'_k \hat{B}_s \) is the sample fit residual.
- Whether this formula is a good estimate depends on the accuracy of control total survey.
Taylor linearization variance estimator (TL)

The Taylor linearization variance estimator is:

\[
\hat{V}_{TL}(\hat{t}_W) = \sum \sum_s \Delta_{kl} w_k e_{ks} w_l e_{ls} + \hat{B}' \hat{V}(\hat{t}_{Cx}) \hat{B}
\]

where

\[
\hat{V} \overset{\text{def}}{=} \hat{V}_{naive} + \hat{V}_{inf}
\]

\[
E_p(\hat{B}' \hat{V}(\hat{t}_{Cx}) \hat{B}) = B' V(t_{Cx}) B + O_p(n^{-2\alpha}).
\]

\[
\hat{V}(\hat{t}_{Cx}) \text{ needs to be specified also from the outside source.}
\]

Taylor Linearization method is likely to be faster than Jackknife methods.
Dever and Valliant, 2010, *Survey Methodology* first used this method in variance estimation of poststratification to population control totals. It is a delete-one jackknife method.

The replicates \( \hat{t}_{Cx(j)} = \hat{t}_{Cx} + c_n \hat{\epsilon}(j) \sqrt{1/(n-1)} \)

where \( \hat{\epsilon}(j) \overset{i.i.d.}{\sim} \text{MVN}(0, \hat{V}(\hat{t}_{Cx})) \) (\( j = 1, 2, \ldots, n \)) and \( c_n = \sqrt{1/(n-1)} \).

The replicates of \( \hat{t}_{yL} \): \( \hat{t}_{yL(j)} = \hat{t}_{Ay(j)} + (\hat{t}_{Cx(j)} - \hat{t}_{Ax(j)})' \hat{B}_s(j) \)

The MVNJ variance estimator is:

\[
\hat{V}_{MVNJ}(\hat{t}_{yW}) = \frac{n-1}{n} \sum_{j=1}^{n} (\hat{t}_{yL(j)} - \hat{t}_{yL})^2.
\]

\[
E_p\{\hat{V}_{MVNJ}\} = E_p[\frac{n-1}{n} \sum_{j=1}^{n} (\hat{t}_{yL(j)}^* - \hat{t}_{yL})^2] + B' V(\hat{t}_{Cx}) B + O_p(n^{-2\alpha})
\]
Fuller two-phase jackknife after Taylor linear. (F2TL)

\[ \hat{t}_{yL} = \sum_s d_k a_{ks} E_k + \hat{t}'_{Cx} B \] and \[ \hat{\theta} - \hat{t}'_{Cx} B = O_p(n^{-1}), \] where \[ \hat{\theta} = \hat{t}'_{Cx} \hat{B}_s. \]

Fuller jackknife method is used to estimate \( V(\hat{\theta}). \)

Let \( \hat{V}(\hat{t}_{Cx}) \) be \( m \times m \) matrix, and \( \lambda_1, \lambda_2,...,\lambda_m \) be its eigenvalues with \( q_1, q_2,...,q_m \) their corresponding eigenvectors.

The replicates are: \( \hat{t}_{Cx(j)} = \hat{t}_{Cx} + c_m \lambda_j^{1/2} q_j \) and \( \hat{\theta}(j) = \hat{t}'_{Cx(j)} \hat{B}_s(j) \), where \( c_m = (m-1)^{-1/2} m^{1/2}. \)

Then the F2TL variance estimator is:

\[ \hat{V}_{F2TL}(\hat{t}_{yW}) = \hat{V}_{naive}(\hat{t}_{yW}) + \frac{m-1}{m} \sum_{j=1}^{m} (\hat{\theta}(j) - \hat{\theta})^2. \]

\( E_p(\hat{V}_{F2TL}) \) is equal to \( E_p(\hat{V}_{naive}) + E_p[\frac{m-1}{m} \sum_{j=1}^{n} (\hat{\theta}^*_j - \hat{\theta})^2] + B' V(\hat{t}_{Cx}) B + O_p(n^{-2\alpha}). \)
Simulation procedure

1. SDR 2010 is chosen as our population. The population contains 27297 individuals, which were divided by us into 54 clusters.

2. At first stage np=30 PSU’s ($U_1$, $U_2$, ..., $U_{30}$) are sampled without replacement out of 54 clusters.

3. Secondly, from within each PSU sampled, we selected 50 individuals using simple random sampling without replacement.

4. Choose salary as the parameter we want to estimate and then choose m=20 auxiliary variables ($X_1$, $X_2$, ..., $X_{20}$).

5. For each $X_i$, calculate the PSU totals for each sampled PSU: ($\hat{t}_{i1}$, $\hat{t}_{i2}$, ..., $\hat{t}_{i30}$)

6. Estimate population totals of $X_i$ using PSU totals, then consider it as estimated control totals.

7. Calculate the calibrated estimator and its variance estimation with four methods mentioned above.
## Simulation Result

<table>
<thead>
<tr>
<th>Variance estimator</th>
<th>RBVE (%)</th>
<th>Cover (%)</th>
<th>MeanSE</th>
<th>StdSE</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Simulation 1. np=20, linear calibration.</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>HT</td>
<td>10.40</td>
<td>94.5</td>
<td>277,002</td>
<td>23,373</td>
</tr>
<tr>
<td>Naive</td>
<td>-96.31</td>
<td>30.4</td>
<td>49,782</td>
<td>10,246</td>
</tr>
<tr>
<td>TL</td>
<td>5.88</td>
<td>95.4</td>
<td>271,200</td>
<td>23,739</td>
</tr>
<tr>
<td>MVNJ</td>
<td>9.71</td>
<td>93.7</td>
<td>272,788</td>
<td>48,803</td>
</tr>
<tr>
<td>F2TL</td>
<td>6.60</td>
<td>96.0</td>
<td>271,668</td>
<td>28,489</td>
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<tr>
<td><strong>Simulation 2. np=30, raking.</strong></td>
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<td></td>
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<tr>
<td>HT</td>
<td>9.56</td>
<td>95.2</td>
<td>190,940</td>
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<tr>
<td>Naive</td>
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<td>31.5</td>
<td>37,586</td>
<td>5,388</td>
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<tr>
<td>TL</td>
<td>4.44</td>
<td>94.1</td>
<td>186,402</td>
<td>11,032</td>
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<tr>
<td>MVNJ</td>
<td>7.46</td>
<td>94.4</td>
<td>187,470</td>
<td>27,049</td>
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<tr>
<td>F2TL</td>
<td>4.92</td>
<td>94.4</td>
<td>186,537</td>
<td>15,280</td>
</tr>
</tbody>
</table>
The CEEC is certainly a reasonable estimate when the population control totals are unknown. In simulation 2, the estimates are close to the true values.

Overall, all the improved variance estimators we give are acceptable in simulation 2. They certainly mitigate the bias of the naive estimator.
Future plans
Thanks!

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