Bayesian Estimation Under Informative Sampling with Unattenuated Dependence

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Inference from Survey Samples

- Goal of Analyst: perform inference about a finite population generated from an unknown model, P_0.
- Data Collected: from under a complex sampling design distribution, P_ν
  - Probabilities of inclusion are often associated with the variable of interest (purposefully)
  - Sampling designs are "informative": the balance of information in the sample $\neq$ balance in the population.
- Biased Estimation: estimate P_0 without accounting for P_ν.

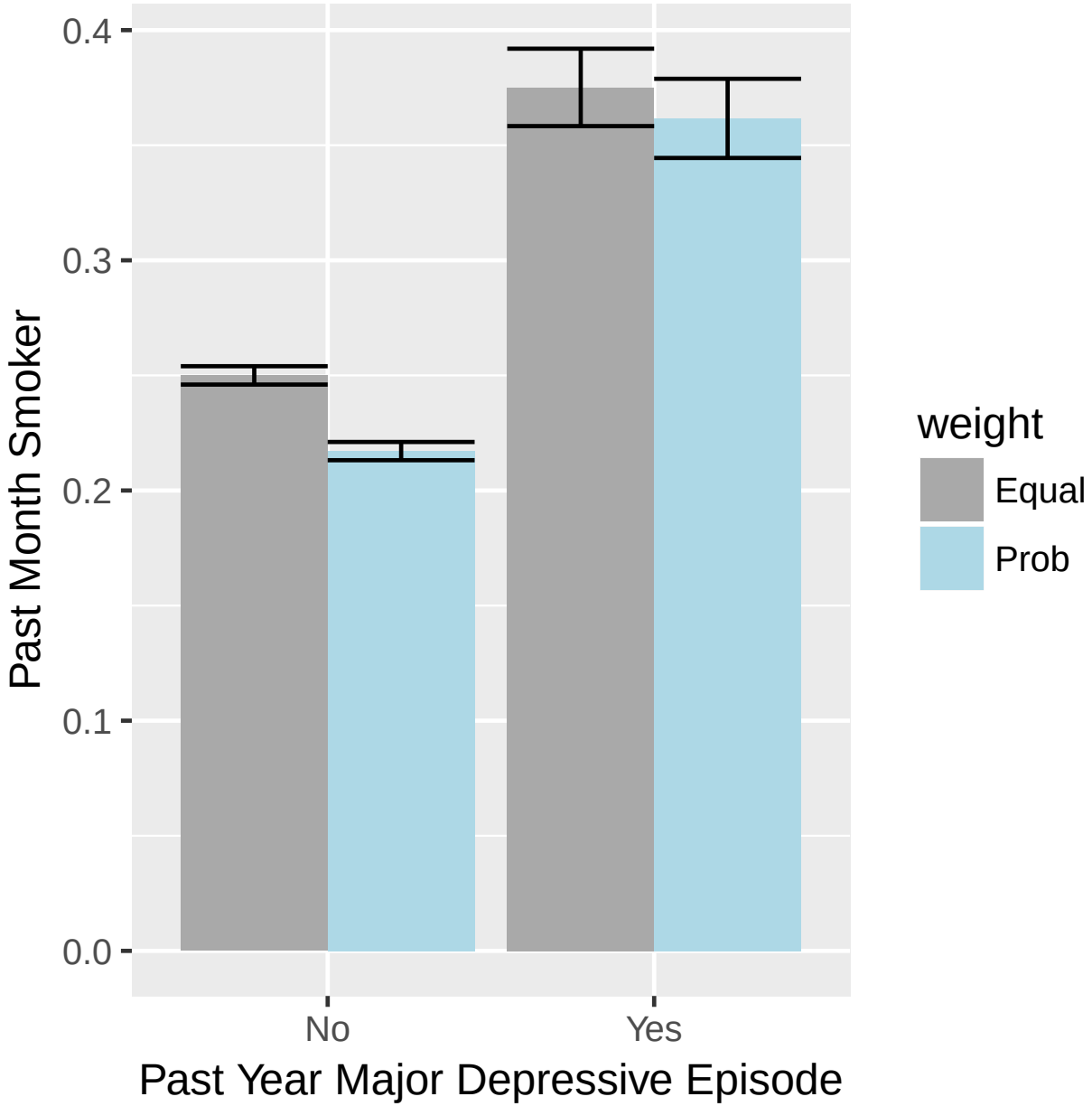
What does it mean for a Design to be Informative?

- Consistency Theory (estimators collapsing to true P_0)
  - Empashize use of inverse probability weights, $w_i = 1/\pi_i$ using marginal (individual) probabilities of being selected
  - Weights are used as plug-in corrections (Savitsky and Toth, 2016) or co-modelled (Novelo and Savitsky, 2017)
  - Sampled elements assumed to become more and more independent with larger sample/population sizes ($\pi_{ij} \rightarrow \pi_i \pi_j$)
- Estimation in Practice
  - Weighting, stratification, and clustering are all incoporated.
  - Variances estimated by aggregating up to the (asymptotically) independent units (cluster ID's or primary sampling units (PSU's).
  - Individual units aren't assumed to be sampled independently in general ($\pi_{ij} \neq \pi_i \pi_j$).

Motivating Example: The National Survey on Drug Use and Health (NSDUH)

- Sponsored by Substance Abuse and Mental Health Services Administration (SAMHSA)
  - Primary source for: alcohol, illicit drug, substance use disorder, mental health issues and their co-occurrence
  - Civilian, non-institutionalized population for 12 and older in the US.
- Multistage, state-based sampling design (Morton et al., 2016)
  - Early stages defined by geography: Select dwelling units (DU's) nested within census block groups (PSU's)
  - Geographic units stratified 'implicitly' via sorting on socioeconomic indicators and selected proportional to compositive population size measure (Systematic PPS).
  - DU's selected within segment via random starting point and selecting every k^{th} unit (Systematic)
  - Individuals (0,1, or 2), selected jointly as pairs with different probabilities assigned to different age-pair combinations (Unequal joint selection of pairs)

NSDUH: Depression and Smoking (12 and older, 2014)



Motivating Example: NSDUH (continued)

- Smoking and depression have the potential to cluster geographically and within dwelling units, since related demographics (age, urbanicity, education, etc) may cluster.
- Current literature is silent on the issue of non-ignorable clustering that may be informative (i.e. related to the response of interest).
- We present conditions for asymptotic consistency for designs like the NSDUH with common features such as:
  - Cluster sampling, selecting only one unit per cluster, or selecting multiple individuals from a dwelling unit.
  - Population information used to sort sampling units along gradients.

Estimation Methods to Account for Informative Sampling

- First-order weights and likelihood
  - Plug-in to exponentiate likelihood (Savitsky and Toth, 2016)
  - Joint modeling of weights and outcome (Novelo and Savitsky, 2017)
- Second-order (pairwise weights) and composite likelihood
  - Clever way to perform hierarchical modelling when design and population model have same structure (Yi et al., 2016)
  - Only beats first-order weights under specific conditions for population model and sample (Williams and Savitsky, 2017)
  - E.g: conditional behavior of spouse-spouse pairs within households AND differential selection of pairs of individuals within a household related to outcome AND subsetting of sample based on information only available in the sample (pair relationship)
- For simplicity, we focus on the *first-order* weights using the *plug-in* method

The Pseudo-Posterior Estimator (Savitsky and Toth, 2016)

The plug-in estimator for posterior density under the analyst-specified model for $\boldsymbol{\lambda} \in \Lambda$ is

$$\hat{\pi}\left(\boldsymbol{\lambda}|\mathbf{y}_o, \tilde{\mathbf{w}}\right) \propto \left[\prod_{i=1}^{n} p\left(y_{o,i}|\boldsymbol{\lambda}\right)^{\tilde{w}_i}\right] \pi\left(\boldsymbol{\lambda}\right), \tag{1}$$

- pseudo-likelihood: $\prod_{i=1}^{n} p\left(y_{o,i}|\boldsymbol{\lambda}\right)^{\tilde{w}_i}$
- prior: $\pi\left(\boldsymbol{\lambda}\right)$
- values $\mathbf{y}_o$ and sampling weights $\{\tilde{\mathbf{w}}\}$ for individuals observed in sample

Consistency Conditions (Savitsky and Toth, 2016)

Based on Empirical Process functionals (Ghosal and van der Vaart, 2007). Population U_ν of size N_ν growing with index $\nu \uparrow \infty$.

(A1) (Local entropy condition - Size of model)

(A2) (Size of space)

(A3) (Prior mass covering the truth)

(A4) (Non-zero Inclusion Probabilities)

$$\sup_{\nu} \left[\frac{1}{\min_{i \in U_\nu} |\pi_{\nu i}|} \right] \leq \gamma, \text{ with } P_0 - \text{probability 1.}$$

(A6) (Constant Sampling fraction)
For constant $f \in (0, 1)$,

$$\limsup_{\nu} \left| \frac{n_\nu}{N_\nu} - f \right| = \mathcal{O}(1), \text{ with } P_0 - \text{probability 1.}$$

Consistency Conditions (NEW)

(A5.1) (Growth of dependence is restricted)
Binary partition $\{S_{\nu 1}, S_{\nu 2}\}$ of the set of all pairs $S_\nu = \{\{i,j\} : i \neq j \in U_\nu\}$ such that

$$\limsup_{\nu \uparrow \infty} |S_{\nu 1}| = \mathcal{O}(N_\nu),$$

and

$$\limsup_{\nu \uparrow \infty} \max_{i,j \in S_{\nu 2}} \left| \frac{\pi_{\nu ij}}{\pi_{\nu i}\pi_{\nu j}} - 1 \right| = \mathcal{O}\left(N_\nu^{-1}\right), P_0 - \text{probability 1}$$

such that for some constants, $C_4, C_5 > 0$ and for N_ν sufficiently large,

$$|S_{\nu 1}| \leq C_4 N_\nu,$$

and

$$N_\nu \sup_{\nu} \max_{i,j \in S_{\nu 2}} \left| \frac{\pi_{\nu ij}}{\pi_{\nu i}\pi_{\nu j}} - 1 \right| \leq C_5,$$

(A5.2) (Special Case: Dependence restricted to countable blocks of bounded size)

Theorem (Updated)

Theorem

Suppose conditions hold. Then for sets $\mathcal{P}_{N_\nu} \subset \mathcal{P}$, *constants,* $K > 0$, *and* M *sufficiently large,*

$$\mathbb{E}_{P_0,P_\nu}\Pi^\pi\left(P : d^\pi_{N_\nu}(P,P_0) \geq M\xi_{N_\nu}|\mathbf{X}_1\delta_{\nu 1},\ldots,\mathbf{X}_{N_\nu}\delta_{\nu N_\nu}\right) \leq$$
$$\frac{16\gamma^2\left[\gamma\mathbf{C_2}+C_3\right]}{\left(Kf+1-2\gamma\right)^2 N_\nu\xi^2_{N_\nu}} + 5\gamma\exp\left(-\frac{Kn_\nu\xi^2_{N_\nu}}{2\gamma}\right), \tag{2}$$

which tends to 0 *as* $(n_\nu, N_\nu) \uparrow \infty$.

Note that $C_2 = C_4 + 1$. When $|S_{\nu 1}| \downarrow 0$ then $C_2 = 1$, leading to main result in Savitsky and Toth (2016).

Simulation Examples: Two (Three) Parameter Logistic Regression

- $$y_i \mid \mu_i \overset{\text{ind}}{\sim} Bern\left(F_l^{-1}(\mu_i)\right), \ i = 1, \ldots, N$$
- $$\boldsymbol{\mu} = -1.88 + 1.0\boldsymbol{x}_1 + 0.5\boldsymbol{x}_2$$
- The x_1 and x_2 distributions are $\mathcal{N}(0, 1)$ and $\mathcal{E}(r = 1/5)$ with rate r
- Size measure used for sample selection is $\tilde{\boldsymbol{x}}_2 = \boldsymbol{x}_2 - \min(\boldsymbol{x}_2) + 1$, *but* neither $\tilde{\boldsymbol{x}}_2$ or $\boldsymbol{x}_2$ are available to the analyst.

Simulation Example: Three-Stage Sample

Area (PPS), Household (Systematic, sorting by Size), Individual (PPS)

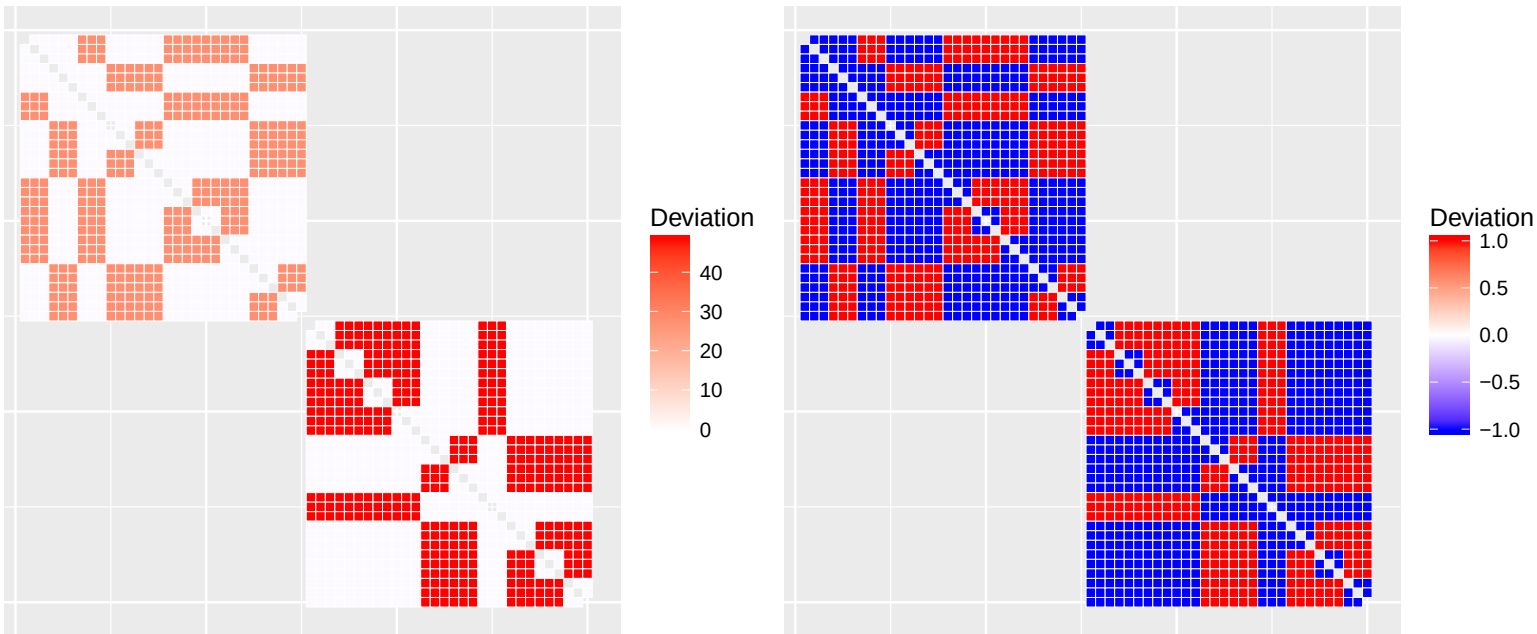


Figure: Factorization matrix $(\pi_{ij}/(\pi_i\pi_j) - 1)$ for two PSU's. Magnitude (left) and Sign (right). Systematic Sampling ($\pi_{ij} = 0$). Clustering and PPS sampling ($\pi_{ij} > \pi_i\pi_j$). Independent first stage sample ($\pi_{ij} = \pi_i\pi_j$)

Simulation Example: Three-Stage Sample (Cont)

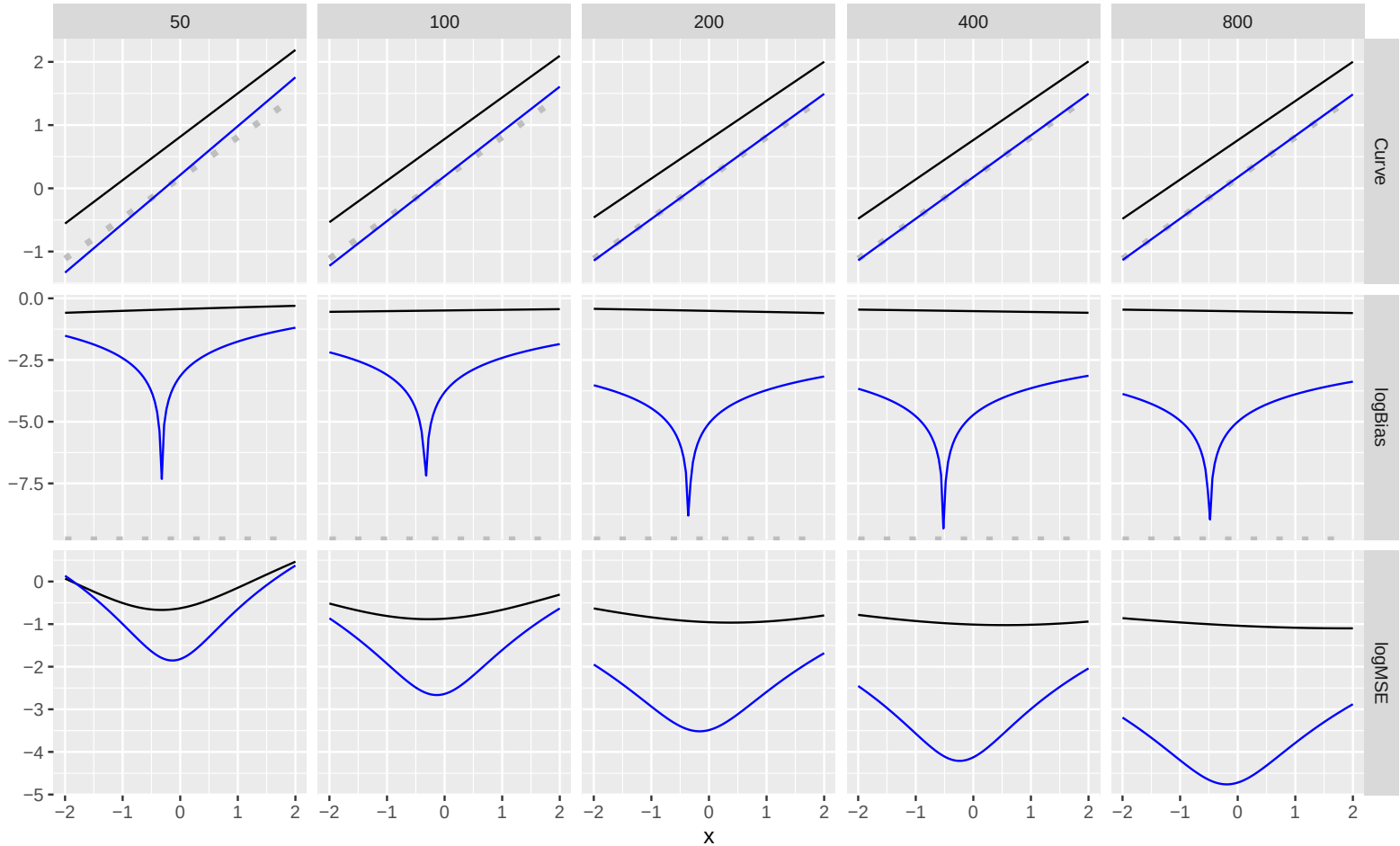


Figure: The marginal estimate of $\mu = f(x_1)$. **population curve**, sample with **equal weights**, and **inverse probability weights**. Top to bottom: estimated curve, log of BIAS, log MSE. Left to right: sample size (50 to 800).

Simulation Example: Sorted Partitions

- Sort population by $\tilde{\boldsymbol{x}}_2$
- Partition the population U into a "high" (U_1) group with the top $N/2$ and a "low" (U_2) group with the bottom $N/2$
- Only two possible samples of size $N/2$: U_1 and U_2.
- Assume an equal probability of selection of $1/2$.
  - $\pi_i = 1/2$, for all $i \in 1, \ldots, N$, and $\pi_{ij} = 1/2$ if $i \neq j \in U_k$, for $k = 1, 2$ and 0 otherwise.
  - Thus, the number of pairwise inclusions probabilities that do not factor ($\pi_{ij} \neq 1/4$) grows at rate $\mathcal{O}\left(N^2\right)$, *violating condition (A5.1)*.
- Alternatively, nest partitions within strata (every 50 sorted units) and sample indepedently across strata
  - factorization for all but $\mathcal{O}(N)$ pairwise inclusion probabilities *statisfies condition (A5.1)*

Simulation Example: Sorted Partitions (Cont)

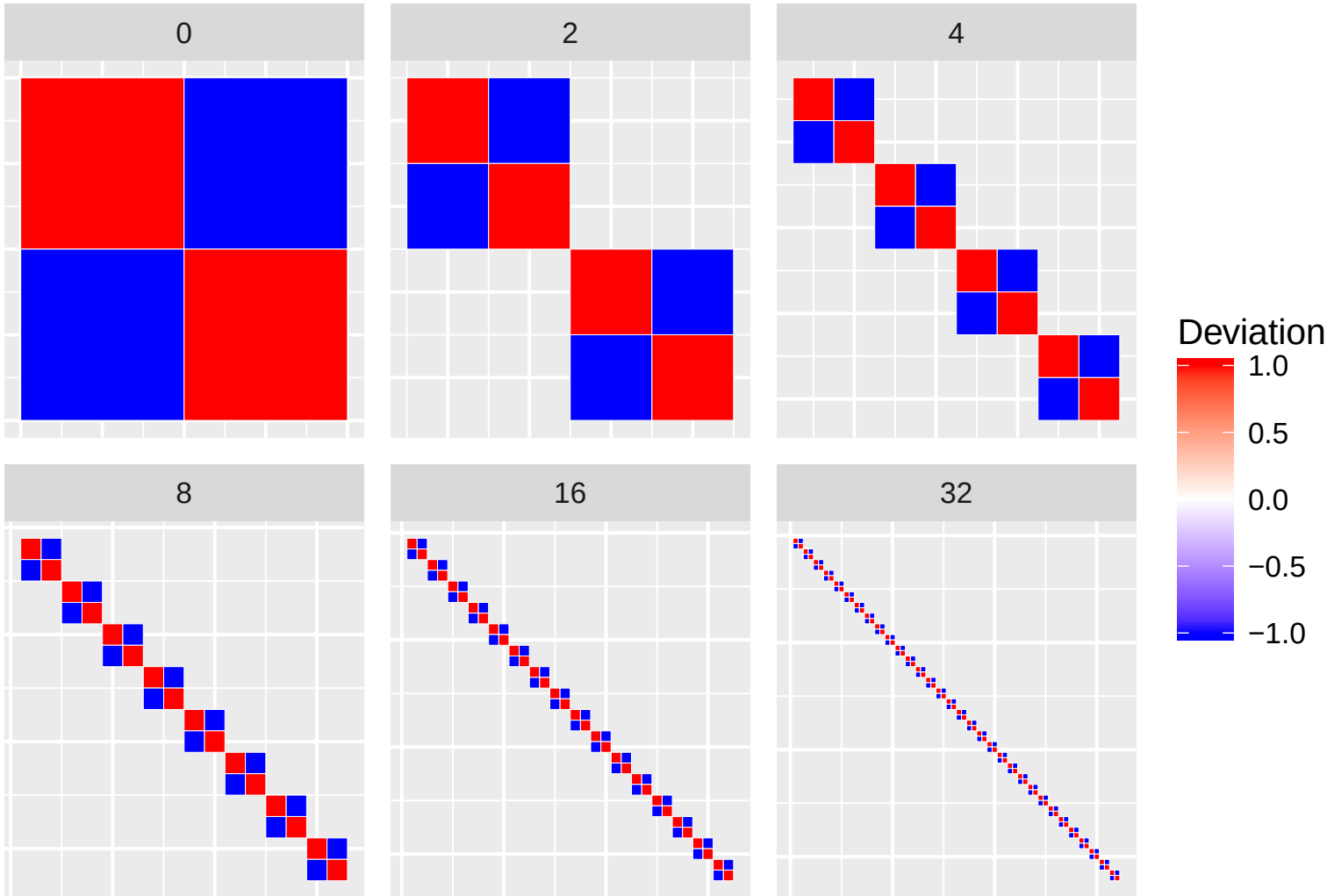


Figure: Matrix $\{i, j\}$ of deviations from factorization $(\pi_{ij}/(\pi_i\pi_j) - 1)$ for an equal probability dyadic partition design by number of strata (0 to 32). Empty cells correspond to 0 deviation (factorization).

Simulation Example: Sorted Partitions (Cont)

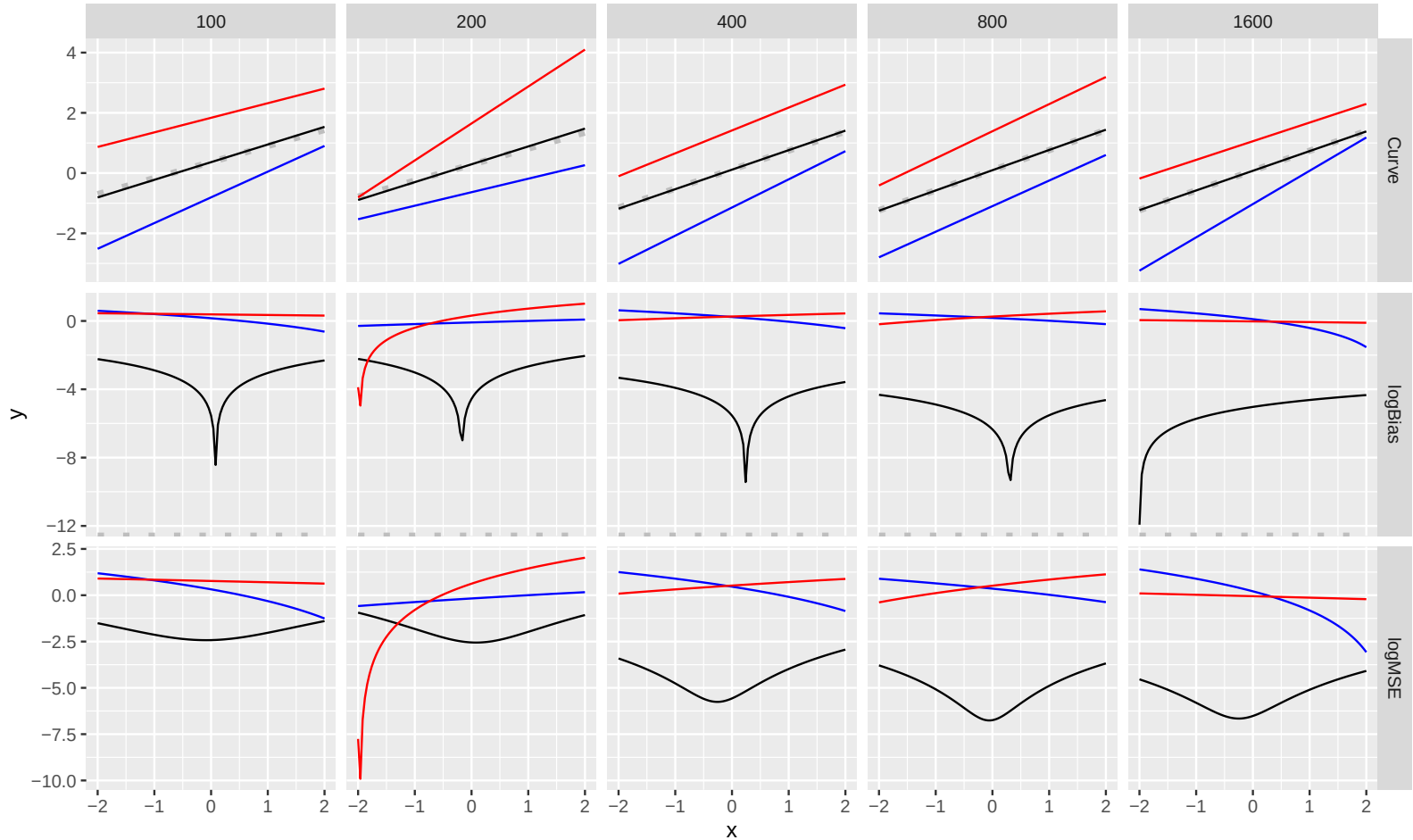


Figure: The marginal estimate of $\mu = f(x_1)$ Compares the **(true) population curve** to binary partition (**high** and **low**) and the **stratified partition** sample. Top to bottom: estimated curve, log of absolute bias, log of mean square error. Left to right: population size (100 to 1600).

Conclusions and Future Research

- **Motivation**: lack of theory to justify consistent estimation for complex, multistage cluster designs such as the NSDUH
  - Approximate or asymptotic factorization of joint sampling probabilities *exclude* these designs
- **Results**: New theoretical conditions now allow for realistic dependence between sampling units.
  - Dependence between units within a cluster is *unrestricted* if **cluster size is bounded** and **dependence between clusters attenuates**. This dependence can be positive (joint selection) or negative (mutual exclusion).
  - Units sorted along a gradient can now be fully justified if the sampling occurs independently **across strata or within clusters**.
- **Current Research**:
  - Impact of survey design on posterior interval estimation.
  - Correcting for mispecification of sampling design efficiency (effective sample size)

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